

# RATIONAL TORUS-EQUIVARIANT STABLE HOMOTOPY IV: THICK TENSOR IDEALS AND THE BALMER SPECTRUM FOR FINITE SPECTRA

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ABSTRACT. We classify thick tensor ideals of finite objects in the category of rational torus-equivariant spectra, showing that they are completely determined by geometric isotropy.

This is essentially equivalent to showing that the Balmer spectrum is the set of closed subgroups under cotoral inclusion.

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## 1. INTRODUCTION

1.A. **Context.** We are interested in the structural properties of the category of  $G$ -equivariant cohomology theories for a compact Lie group  $G$ . To start with, the category is tensor-triangulated, and it is the homotopy category of the category of  $G$ -spectra. To see the broad features of this category we restrict attention to cohomology theories whose values are rational vector spaces: a  $G$ -equivariant cohomology theory with values in rational vector spaces is represented by a  $G$ -spectrum with rational homotopy groups.

The category of rational  $G$ -spectra has a very rich structure. The case when  $G$  is a torus has been studied extensively [11, 12, 13], and there is a complete algebraic model for rational  $G$ -spectra [15]. In this paper we study the structural features (as a tensor triangulated category) of the homotopy category of finite rational  $G$ -spectra when  $G$  is a torus. From now on the standing assumption is that all spectra are rational and until Section 9 the group  $G$  is a rank  $r$  torus. In Sections 9 and 10 we describe the consequences for more general groups  $G$ .

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I am grateful to B.Sanders for conversations about this work.

**1.B. Overview.** In fact, we give a classification of the finitely generated thick tensor ideals of finite rational  $G$ -spectra for a torus  $G$ . This is in terms of traditional invariants of transformation groups, namely fixed points and Borel cohomology.

Recall that the geometric isotropy of a  $G$ -spectrum  $X$ ,

$$\mathcal{I}_g(X) = \{K \mid \Phi^K X \not\simeq_1 0\}$$

is the collection of closed subgroups  $K$  for which the geometric fixed points  $\Phi^K X$  are non-equivariantly essential. We say that  $L$  is *cotoral* in  $K$  if  $L$  is a normal subgroup of  $K$  and  $K/L$  is a torus.

**Theorem 1.1.** (i) *If  $X$  is a finite rational  $G$ -spectrum then  $\mathcal{I}_g(X)$  is closed under passage to cotoral subgroups and has only finitely many maximal elements.*

(ii) *If  $X$  and  $Y$  are finite rational  $G$ -spectra with  $\mathcal{I}_g(X) = \mathcal{I}_g(Y)$  then  $X$  and  $Y$  generate the same thick tensor ideal.*

The closure under passage to cotoral subgroups comes from the classical Borel-Hsiang-Quillen Localization Theorem. The rest requires more detailed analysis.

One may also reformulate this in terms of the Balmer spectrum of the category of rational  $G$ -spectra. Indeed, the Balmer primes are in 1:1 correspondence with the closed subgroups, containment of primes corresponds precisely to cotoral inclusion and the Balmer support is precisely the geometric isotropy.

It is a remarkable vindication of the Balmer spectrum that for the groups considered so far it captures the space of subgroups and cotoral inclusions, and even the  $f$ -topology of [7]. This can be put down to the fact that both the Balmer spectrum and the analysis of rational  $G$ -spectra are principally based on the Localization theorem and the calculation of the rational Burnside ring. For some of the more delicate arguments it is convenient to use the very structured packaging of these ingredients used in the algebraic model of rational  $G$ -spectra [11, 12, 15].

The outline of the argument is as follows. First, we note that rationally all *natural* cells  $G/K_+$  are finitely built from certain *basic* cells  $\sigma_K$  ([11], see Subsection 3.C). Any  $X$  can be built from basic cells  $\sigma_K$  for  $K$  in the cotoral closure of the geometric isotropy of  $X$ . This shows that the thick tensor ideal corresponding to any admissible collection of subgroups  $A$  is generated by the wedge of  $\sigma_K$  with  $K$  maximal in  $A$ . To complete the proof one must show that the thick tensor ideal generated by  $X$  contains  $\sigma_K$  for  $K$  maximal in the geometric isotropy of  $X$ ; this is the point where use of the algebraic model plays a central role.

**1.C. Models.** We are interested in the tensor-triangulated homotopy category of rational  $G$ -spectra for a torus  $G$ , and in the tensor-triangulated derived category of an abelian category  $\mathcal{A}(G)$ .

A Quillen equivalence

$$G\text{-spectra}/\mathbb{Q} \simeq d\mathcal{A}(G)$$

is proved in [15], where  $d\mathcal{A}(G)$  consists of differential graded objects of  $\mathcal{A}(G)$ . It follows that there is a triangulated equivalence

$$Ho(G\text{-spectra}/\mathbb{Q}) \simeq D(\mathcal{A}(G)).$$

For the circle group the Quillen equivalence is shown to be monoidal in [3], so the triangulated equivalence preserves the tensor structure. This is expected to be the case for a general torus  $G$ , but that is work in progress.

For the present therefore, we need separate treatments for the two tensor-triangulated categories  $Ho(G\text{-spectra}/\mathbb{Q})$  and  $D(\mathcal{A}(G))$ . This is irritating, but the arguments we will give apply equally well to both examples: they only require the existence of a homology functor from the tensor triangulated category to  $\mathcal{A}(G)$ , and the existence of an Adams spectral sequence based on it.

Because of this distinction, we have broken the argument into two parts for purely expository reasons. In the first part (Sections 2 and 3) it is easy to give the proof in the conventional language of  $G$ -spectra, so we give it before describing the category  $\mathcal{A}(G)$  in Section 4. The transcription of the first part from the homotopy category of  $G$ -spectra to  $D(\mathcal{A}(G))$  is immediate. In the second part (Sections 5 to 7) it is convenient to use the formal structure of  $\mathcal{A}(G)$  so we give the argument in  $D(\mathcal{A}(G))$ . Nonetheless, the arguments with the *abelian* category  $\mathcal{A}(G)$  apply simultaneously to the homology of an object of  $d\mathcal{A}(G)$  and to  $\pi_*^{\mathcal{A}}(X)$  for a  $G$ -spectrum  $X$ , and the triangulated arguments apply to the equivalent categories. The tensor arguments can be conducted in either one of the two structures.

Readers who feel nervous riding two horses may wish to fix their attention on  $\mathcal{A}(G)$ .

**1.D. Contents.** In Section 2 we recall some ideas from the classical theory of transformation groups. In Section 3 we introduce a structure of cells and isotropy for studying rational spectra efficiently. In Section 4 we briefly outline the algebraic model for rational  $G$ -spectra from [11, 12, 13]. Section 5 is the core of the paper, and contains the key facts showing that all thick tensor ideals are determined by the basic cells in them. In Section 6 we recall the definitions of Balmer primes, and show that from this point of view, the work so far has been a theory of support. In Section 7 we use the classification of finitely generated thick tensor ideals to show that all primes are the obvious ones corresponding to vanishing of geometric fixed points at a closed subgroup.

We then turn to complementary results. In Section 8 we show that how great an effect the ideal condition has; whereas the unit  $S^0$  gives everything as a thick tensor ideal, we identify explicitly the proper thick subcategory it generates in the category of finite semifree  $T$ -spectra. Finally, we turn to some other groups. In Section 9 we deduce from the case of the circle what happens for  $O(2)$  and  $SO(3)$ , and in Section 10 we consider toral spectra for general groups  $G$ , and in particular show that the Balmer spectrum is obtained from that of the maximal torus by passing to the quotient under the Weyl group action (the Going Up and Going Down arguments may be of interest in themselves).

**1.E. Notation.** Given a partially ordered set (*poset*)  $X$  and a subset  $A \subseteq X$ , we write  $\Lambda_{\leq}(A) = \{b \in X \mid b \leq a \in A\}$  for the downward closure of  $A$ .

We will consider two partial orderings on the set of all closed subgroups of a compact Lie group  $G$ . We indicate containment of subgroups by  $K \subseteq H$ , and refer to this as the *classical* ordering. Accordingly  $\Lambda_{cl}(K)$  consists of all closed subgroups of  $K$ . The more significant ordering in this paper is that of *cotoral inclusion*. We write  $K \leq H$  if  $K$  is normal in  $H$  and  $H/K$  is a torus. The importance of this ordering arises from the Localization Theorem. The set  $\Lambda_{ct}(K)$  consists of all closed subgroups cotoral in  $K$ .

Given an object  $X$  of a triangulated category, we write  $\text{thick}(X)$  for the thick subcategory generated by  $X$  (i.e., the set of objects that can be built from  $X$  by completing triangles and taking retracts). We write  $\text{thick}_{\otimes}(X)$  for the thick tensor ideal generated by  $X$  (i.e., the

set of objects that can be built from  $X$  by completing triangles, tensoring with an arbitrary object and taking retracts).

**1.F. Conventions.** All homology and cohomology will be reduced and have rational coefficients.

A  $G$ -equivalence of  $G$ -spectra will be denoted  $X \simeq Y$ . A non-equivariant equivalence of  $G$ -spectra will be denoted  $X \simeq_1 Y$  for emphasis.

The natural extension to  $G$ -spectra of the  $H$ -fixed point functor on based  $G$ -spaces is the geometric  $H$ -fixed point functor:

$$\Sigma^\infty(P^H) \simeq \Phi^H(\Sigma^\infty P).$$

Since we routinely omit the suspension spectrum functor, we may write  $\Phi^H P$  for the  $H$ -fixed point space of a based space  $P$  to avoid ambiguity.

## 2. THE LOCALIZATION THEOREM

We revisit some basic facts from transformation groups in our language. The basic tool is Borel cohomology,  $H_G^*(X) := H^*(EG \times_G X, EG \times_G pt) = H^*(EG_+ \wedge_G X)$ .

The idea is that for finite spectra, geometric isotropy is determined by Borel cohomology. It then follows from the Localization Theorem that the geometric isotropy is closed under passage to cotoral subgroups.

**Lemma 2.1.** *If  $K$  is a torus and  $X$  is a finite  $K$ -CW-complex, then  $X$  is non-equivariantly contractible if and only if  $H_K^*(X) = 0$ .*

**Proof:** First, if  $X$  is simply connected, the Hurewicz theorem shows  $X \simeq *$  if and only if  $H_*(X) = 0$ . Since  $X$  is finite, that is equivalent to  $H^*(X) = 0$ .

Next, we have a fibration  $X \rightarrow EK \times_K X \rightarrow BK$  so the Serre spectral sequence shows that if  $H^*(X) = 0$  then also  $H_K^*(X) = 0$ . Conversely, with unreduced cochains the Eilenberg-Moore theorem gives

$$C^*(X) \simeq C^*(EK \times_K X) \otimes_{C^*(BK)} \mathbb{Q},$$

(where the tensor product is derived). This shows that  $H_K^*(X) = 0$  implies  $H^*(X) = 0$ .  $\square$

It follows that  $\mathcal{I}_g(X)$  can be detected from Borel cohomology of fixed points.

**Corollary 2.2.** *If  $X$  is finite,  $K \in \mathcal{I}_g(X)$  if and only if  $H_{G/K}^*(\Phi^K X) \neq 0$ .*  $\square$

For us the fundamental fact is the following consequence of the Localization Theorem.

**Proposition 2.3.** *If  $X$  is finite then  $\mathcal{I}_g(X)$  is closed under passage to cotoral subgroups.*

**Proof:** The Localization Theorem states that if  $K$  is a torus and  $X$  is a finite  $K$ -CW-complex then

$$H_K^*(X) \rightarrow H_K^*(\Phi^K X) = H^*(BK) \otimes H^*(\Phi^K X)$$

becomes an isomorphism when the multiplicatively closed set  $\mathcal{E}_K = \{e(W) \mid W^K = 0\}$  of Euler classes  $e(W) \in H^{|W|}(BK)$  is inverted. The proof uses the fact that the space

$$S^{\infty V(K)} = \bigcup_{W^K=0} S^W,$$

has  $H$ -fixed points  $S^0$  if  $H \supseteq K$  and contractible otherwise. Accordingly, we have a  $G$ -equivalence

$$X \wedge S^{\infty V(K)} \simeq \Phi^K X \wedge S^{\infty V(K)}.$$

It follows that if  $H^*(\Phi^K X) \neq 0$  then also  $H_K^*(X) \neq 0$ . The conclusion follows from Lemma 2.2.  $\square$

### 3. BASIC CELL COMPLEXES

A distinctive feature of working rationally is that there are many idempotents in the rational Burnside ring of a finite group. We follow through the implications of this for cell structures.

Integrally, the relevant cells are homogeneous spaces  $G/K_+$ , and the relevant ordering of subgroups is classical containment. Rationally, the splitting of Burnside rings means that the relevant cells are certain basic cells  $\sigma_K$  (a retract of  $G/K_+$  introduced below) and the relevant ordering of subgroups is cotoral inclusion.

We begin by running through one approach to equivariant cell complexes, and then introduce basic cells and follow the same pattern to give the rational analysis in terms of basic cells.

**3.A. Unstable classical recollections.** Classically, we are used to the idea that based  $G$ -spaces  $P$  are formed from cells  $G/K_+$ . The classical unstable isotropy is defined by

$$\mathcal{I}'_{un}(P) = \{K \mid P^K \neq pt\};$$

it is not homotopy invariant, but it does have the obvious property that it is closed under passage to subgroups.

It is natural to move to a homotopy invariant notion

$$\mathcal{I}_{un}(P) = \{K \mid P^K \not\approx *\}.$$

This notion fits well with the cells we use, since

$$\mathcal{I}_{un}(G/K_+) = \{L \mid L \subseteq K\} = \Lambda_{cl}(K).$$

Note that we are linking notions of cell and isotropy with a partial order on subgroups.

The homotopy invariant version of unstable isotropy may not be closed under passage to subgroups, so that we only know that  $X$  is equivalent to a complex constructed from cells  $G/K_+$  with  $K \in \Lambda_{cl}\mathcal{I}_{un}(T)$ , where  $\Lambda_{cl}$  indicates that we take the closure under the classical order (i.e., under containment). This can be proved by killing homotopy groups, or by the method described in the next subsection for the stable situation.

**3.B. Stable classical recollections.** Moving to the stable world, for a  $G$ -spectrum  $X$  we have the stable isotropy

$$\mathcal{I}_g(X) = \{K \mid \Phi^K X \not\approx_1 0\},$$

homotopy invariant by definition. Evidently since geometric fixed points extend ordinary fixed points on spaces,

$$\mathcal{I}_{un}(P) \supseteq \mathcal{I}_g(\Sigma^\infty P),$$

and if  $P$  can be constructed from cells in  $A$  then  $\Sigma^\infty P$  can be constructed from stable cells in  $A$ .

**Remark 3.1.** In [5] and the author's subsequent work this was called *stable isotropy* to emphasize that stability is the main change from  $\mathcal{I}_{un}$ . The corresponding notion for categorical fixed points does not seem to be useful, so this caused no confusion.

The name of 'geometric isotropy' from [17] seems to have acquired currency, and the symmetry between 'stable' and 'unstable' does not seem sufficient to overturn this advantage.

The attraction of geometric isotropy and geometric fixed points arises from the fact that their properties are familiar from the category of based spaces. Perhaps the most important instance is that of the Geometric Fixed Point Whitehead Theorem, and we state it here because it is fundamental to our approach. The result is well known to all users of geometric fixed points, and is usually deduced using isotropy separation to see that that an equivalence in all geometric fixed points is an equivalence in all categorical fixed points.

**Lemma 3.2.** (Geometric Fixed Point Whitehead Theorem) *A map  $f : X \rightarrow Y$  of  $G$ -spectra is an equivalence if  $\Phi^K f : \Phi^K X \rightarrow \Phi^K Y$  is a non-equivariant equivalence for all closed subgroups  $K$ .  $\square$*

Next, we note that

$$\mathcal{I}_g(G/K_+) = \Lambda_{cl}(K).$$

Any  $G$ -spectrum  $X$  can be constructed from stable cells  $G/K_+$  with  $K \in \Lambda_{cl}\mathcal{I}_g(X)$ . One method is to construct a filtration analogous to the (thickened) fixed point filtration of a space. Simplifying this, if  $\mathcal{F} = \Lambda_{cl}\mathcal{I}_g(X)$  then  $X \wedge \tilde{E}\mathcal{F}$  has trivial geometric fixed points (and is thus contractible by the Geometric Fixed Point Whitehead Theorem). Now  $X \wedge E\mathcal{F}_+$  may be constructed from cells  $G/K_+$  for  $K \in \mathcal{F}$ . In effect we use the result for the special case  $E\mathcal{F}_+$  together with the fact that  $G/H_+ \wedge G/K_+$  can be constructed from cells  $G/K'_+$  with  $K' \subseteq K$ .

We now see how we can take advantage of the additional flexibility of working rationally.

**3.C. Basic cells.** The classical cell  $G/K_+$  is  $S^0$  induced up from  $K$ , so if the  $K$ -equivariant sphere decomposes, so does  $G/K_+$ . We know  $[S^0, S^0]^K = A(K)$  is the Burnside ring. Rationally this consists of continuous  $\mathbb{Q}$ -valued functions on the space  $\Phi K$  of conjugacy classes of subgroups of finite index in its normalizer [4]. When  $K$  has identity component a torus this means  $A(K) \cong \prod_{(\overline{L})} \mathbb{Q}$ , with the product over conjugacy classes of subgroups  $\overline{L}$  of  $\overline{K} = \pi_0(K)$ . Accordingly we obtain one primitive idempotent  $e_{\overline{L}}$  for each conjugacy class. The building blocks are thus the *basic cells*

$$\sigma_K := G_+ \wedge_K e_K S^0.$$

We develop cell structures based on these. The point is that the geometric isotropy of  $\sigma_K$  is smaller than that of  $G/K_+$ .

**Lemma 3.3.** *The geometric isotropy of  $\sigma_K$  consists of all cotoral subgroups of  $K$ :*

$$\mathcal{I}_g(\sigma_K) = \Lambda_{ct}(K).$$

**Proof:** The  $K$ -spectra  $e_L S^0$  for  $L$  a proper subgroup of  $K$  can be constructed with non-fixed  $K$ -cells, so this is true of the cofibre of the inclusion  $i : e_K S^0 \rightarrow S^0$ . Inducing up to  $G$ , the cofibre of  $i : \sigma_K = G_+ \wedge_K e_K S^0 \rightarrow G_+ \wedge_K S^0 \simeq G/K_+$  is constructed from cells

$G/L_+$  with  $L$  a proper subgroup of  $K$ . Thus  $\Phi^K i$  is a non-equivariant equivalence. Since  $\Phi^K G/K_+ = G/K_+ \not\simeq 0$ ,  $K \in \mathcal{I}_g(\sigma_K)$  and hence by Proposition 2.3  $\Lambda_{ct}(K) \subseteq \mathcal{I}_g(\sigma_K)$ .

Conversely, since  $\sigma_K$  is a retract of  $G/K_+$ ,  $\mathcal{I}_g(\sigma_K)$  consists of subgroups of  $K$ , and if  $L \subset K$  is not cotoral in  $K$  then there is an idempotent  $e_L$  orthogonal to  $e_K$  with  $\Phi^L(e_L S^0) = \Phi^L(S^0)$  and hence  $\Phi^L(\sigma_K) = \Phi^L(G_+ \wedge_K e_L e_K S^0) \simeq 0$ .  $\square$

**3.D. Basic detection.** Basic cells play a comparable role to classical cells in that they generate the category and detect equivalences. The smaller isotropy means that we can make slightly stronger statements.

**Lemma 3.4.** *The category of rational  $G$ -spectra is generated by the basic cells.*

**Proof:** It suffices to show that the classical cells  $G/K_+$  are built from the basic cells. The proof is by induction on  $K$  (i.e., we work with the poset of all subgroups ordered by inclusion, and note that there are no infinite decreasing chains).

Since  $G/1_+ = \sigma_1$ , the induction begins. Now suppose  $K$  is non-trivial and that  $G/L_+$  is built from basic cells for proper subgroups  $L$  of  $K$ . Now  $G/K_+$  is a sum of  $\sigma_K$  and the spectra  $G_+ \wedge_K e_L S^0$  for  $L \subset K$ . The spectrum  $G_+ \wedge_K e_L S^0$  is built from cells  $G/M_+$  for  $M \subseteq K$  as in Lemma 3.3 and hence from basic cells by induction.  $\square$

**Remark 3.5.** The proof shows that  $G/K_+$  is built from basic cells  $\sigma_L$  with  $L \subseteq K$ .

There is a useful criterion for vanishing of homotopy in terms of geometric isotropy.

**Lemma 3.6.** (i) *If  $\Lambda_{cl}(K) \cap \mathcal{I}_g(X) = \emptyset$  then  $[G/K_+, X]_*^G = 0$ .*

(ii) *If  $\Lambda_{ct}(K) \cap \mathcal{I}_g(X) = \emptyset$  then  $[\sigma_K, X]_*^G = 0$*

**Proof:** The first statement is immediate from the Geometric Fixed Point Whitehead Theorem, since  $\mathcal{I}_g(\text{res}_K^G X) = \emptyset$ .

For the second, we note

$$[\sigma_K, X]_*^G = [e_K S^0, X]_*^K = [e_K S^0, e_K X]_*^K.$$

By hypothesis  $\mathcal{I}_g(e_K X) = \Lambda_{ct}(K) \cap \mathcal{I}_g(\text{res}_K^G X) = \emptyset$ , so that  $e_K X \simeq_K 0$  by the Geometric Fixed Point Whitehead Theorem.  $\square$

A slightly refined version of the Whitehead Theorem holds in the rational context.

**Proposition 3.7.** *Suppose that  $\mathcal{I}_g(X), \mathcal{I}_g(Y) \subseteq \mathcal{K}$  and  $f : X \rightarrow Y$  induces an isomorphism of  $[\sigma_K, \cdot]_*^G$  for all  $K \in \mathcal{K}$ . Then  $f$  is an equivalence.*

**Proof:** Taking mapping cones, it suffices to show that if  $\mathcal{I}_g(Z) \subseteq \mathcal{K}$  and  $[\sigma_K, Z]_*^G = 0$  for all  $K \in \mathcal{K}$  then  $Z \simeq 0$ .

By Lemma 3.4, it suffices to show  $[\sigma_K, Z]_*^G = 0$  for all  $K$ . There are three cases. If  $K \in \mathcal{I}_g(Z)$  then the vanishing follows from the hypothesis on  $f$ . If  $\Lambda_{ct}(K) \cap \mathcal{I}_g(Z) = \emptyset$ , it follows from Lemma 3.6.

Finally if  $K \notin \mathcal{I}_g(Z)$  but there is a subgroup  $L \in \Lambda_{ct}(K) \cap \mathcal{I}_g(Z)$ , then in view of Proposition 2.3 we may take  $L$  to be finite. Now, from the hypothesis we have

$$0 = [\sigma_L, Z]_*^G = e_L[S^0, Z]_*^L = [S^0, \Phi^L Z]_*.$$

By the Serre spectral sequence we conclude  $H_{H/L}^*(\Phi^L Z) = 0$  for every  $H$  with  $H/L$  a torus. It follows that  $\Lambda_{ct}(K/L) \cap \mathcal{I}_g(\Phi^L Z) = \emptyset$  and  $e_{K/L}\Phi^L Z \simeq 0$ . Hence  $\Phi^K Z = \Phi^{K/L}\Phi^L Z \simeq 0$ .

Combining the three cases,  $\Lambda_{ct}K \cap \mathcal{I}_g Z = \emptyset$ , and therefore by Lemma 3.6  $[\sigma_K, Z]_*^G = 0$  as required.  $\square$

**3.E. Basic structures.** When we work rationally, classical containment of subgroups is replaced by cotoral inclusion. Cells  $G/K_+$  are no longer indecomposable, and we have basic cells  $\sigma_K$ . If  $K$  is a torus then  $\sigma_K = G/K_+$ , but if  $K$  is not a torus then we have a proper inclusion

$$\mathcal{I}_g(\sigma_K) = \Lambda_{ct}(K) \subset \Lambda_{cl}(K) = \mathcal{I}_g(G/K_+).$$

**Lemma 3.8.** *Any  $G$ -spectrum  $X$  can be constructed from basic cells  $\sigma_K$  with  $K$  in  $\Lambda_{ct}(\mathcal{I}_g(X))$ .*

**Proof:** Take  $\mathcal{K} = \Lambda_{ct}\mathcal{I}_g(X)$ . We may construct a map  $p : P \rightarrow X$  so that  $P$  is a wedge of suspensions of basic cells  $\sigma_K$  for  $K \in \mathcal{K}$ , and so that  $p_*$  is surjective on  $[\sigma_K, \cdot]_*^G$  for all  $K \in \mathcal{K}$ . Iterating this, we form a diagram

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & P_0 & & P_1 & & P_2 \end{array}$$

We take  $X_\infty = \text{holim}_{\rightarrow} X_s$ , and note that since  $\sigma_K$  is small for each  $K$ , it follows  $[\sigma_K, X_\infty]_*^G = 0$  for  $K \in \mathcal{K}$ . Since  $\mathcal{I}_g(X_\infty) \subseteq \Lambda_{ct}\mathcal{I}_g(X) = \mathcal{K}$ , it follows from Proposition 3.7 that  $X_\infty \simeq 0$ . Arguing with the dual tower, we see  $X$  can be constructed from cells  $\sigma_K$ : indeed, we define  $X^s$  by the cofibre sequence

$$X^s \rightarrow X \rightarrow X_s.$$

By definition  $X^0 \simeq *$ , and we have

$$\Sigma^{-1}P_s \rightarrow X^s \rightarrow X^{s+1}.$$

Again  $X^\infty = \lim_{\rightarrow s} X^s$ , and since  $X_\infty \simeq 0$ , we see  $X^\infty \simeq X$ .  $\square$

**Remark 3.9.** One can imagine other proofs. One is to construct an analogue of the map  $E\mathcal{F}_+ \rightarrow S^0$  for a family  $\mathcal{F}$ . This is a spectrum  $E\langle\mathcal{K}\rangle$  with geometric isotropy  $\mathcal{K}$  and a map  $E\langle\mathcal{K}\rangle \rightarrow S^0$  which is an equivalence in geometric  $K$ -fixed points for all  $K \in \mathcal{K}$  (one construction follows from the results below). One then mimics the rest of the proof in the classical case.

It then follows that  $X \simeq X \wedge E\langle\mathcal{K}\rangle$ . Now we construct  $E\langle\mathcal{K}\rangle$  out of basic cells  $\sigma_K$  with  $K \in \mathcal{K}$ , and claim that  $G/H_+ \wedge \sigma_K$  can be constructed from basic cells  $\sigma_{K'}$  for  $K'$  cotoral in  $K$ .

It may also be useful to formulate a statement which replaces a classical cell structure by a basic cell structure with basic cells  $\sigma_K$  for  $K$  lying in a cotorally closed collection.

The two natural notions of finiteness for rational spectra coincide.

**Lemma 3.10.** *A rational  $G$ -spectrum is constructed from finitely many basic cells  $\sigma_K$  if and only if it is constructed from finitely many classical cells  $G/K_+$ .*



**Proof:** Since  $\sigma_K$  is a retract of  $G/K_+$ , a basic-finite complex is a finite complex. The standard cell  $G/K_+$  is a basic-finite complex (it is built by basic cells using Lemma 3.4, and then  $G/K_+$  is a retract of a finite basic complex using smallness). Accordingly, any classical-finite complex is basic-finite.  $\square$

#### 4. THE ABELIAN MODEL OF RATIONAL $G$ -SPECTRA

The main theorem of [15] states that there is a Quillen equivalence

$$G\text{-spectra}/\mathbb{Q} \simeq d\mathcal{A}(G)$$

between rational  $G$ -spectra and differential objects of  $\mathcal{A}(G)$ . So far we have worked directly in the category of rational  $G$ -spectra, but for the most delicate part of the proof it is convenient to use the algebraic model.

Accordingly we summarize the definition and properties of the abelian category  $\mathcal{A}(G)$  that we need from [11] including an Adams spectral sequence based on it. The present account is very brief and readers may need to refer to [11] for details. The structures from that analysis will be relevant to much of what we do here.

**4.A. Definition of the category.** First we must construct the category  $\mathcal{A}(G)$ , which is a category of modules over a diagram of rings. For a category  $\mathbf{D}$  and a diagram of  $R : \mathbf{D} \rightarrow \mathbf{Rings}$  of rings, an  $R$ -module is given by a  $\mathbf{D}$ -diagram  $M$  such that  $M(x)$  is an  $R(x)$ -module for each object  $x$  in  $\mathbf{D}$ , and for every morphism  $a : x \rightarrow y$  in  $\mathbf{D}$ , the map  $M(a) : M(x) \rightarrow M(y)$  is a module map over the ring map  $R(a) : R(x) \rightarrow R(y)$ .

The shape of the diagram for  $\mathcal{A}(G)$  is given by the partially ordered set  $\text{Sub}_c(G)$  of connected subgroups of  $G$ . To start with we consider the single graded ring

$$\mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(BG/F),$$

where the product is over the family  $\mathcal{F}$  of finite subgroups of  $G$ . To specify the value of the ring at a connected subgroup  $K$ , we use Euler classes: indeed if  $V$  is a representation of  $G$  we may define  $c(V) \in \mathcal{O}_{\mathcal{F}}$  by specifying its components. In the factor corresponding to the finite subgroup  $F$  we take  $c(V)(F) := c_{|V^F|}(V^F) \in H^{|V^F|}(BG/F)$  where  $c_{|V^F|}(V^F)$  is the classical Euler class of  $V^H$  in ordinary rational cohomology.

The diagram of rings  $\widetilde{\mathcal{O}}_{\mathcal{F}}$  is defined by the following functor on  $\text{Sub}_c(G)$

$$\widetilde{\mathcal{O}}_{\mathcal{F}}(K) = \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}}$$

where  $\mathcal{E}_K = \{c(V) \mid V^K = 0\} \subseteq \mathcal{O}_{\mathcal{F}}$  is the multiplicative set of Euler classes of  $K$ -essential representations. Each of the Euler classes is a finite sum of mutually orthogonal homogeneous terms, and so this localization is again a graded ring. Next we consider the category of modules  $M$  over the diagram  $\widetilde{\mathcal{O}}_{\mathcal{F}}$ . Thus the value  $M(K)$  is a module over  $\mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}}$ , and if  $L \subseteq K$ , the structure map

$$M(L) \rightarrow M(K)$$

is a map of modules over the map

$$\mathcal{E}_L^{-1} \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}}$$

of rings. Note this map of rings is a localization since  $V^L = 0$  implies  $V^K = 0$  so that  $\mathcal{E}_L \subseteq \mathcal{E}_K$ . The category  $\mathcal{A}(G)$  is formed from a subcategory of the category of  $\widetilde{\mathcal{O}}_{\mathcal{F}}$ -modules by adding structure. There are two requirements which we briefly indicate here. Firstly they must be *quasi-coherent*, in that they are determined by their value at the trivial subgroup 1 by the formula

$$M(K) := \mathcal{E}_K^{-1} M(1).$$

The second condition involves the relation between  $G$  and its quotients. Choosing a particular connected subgroup  $K$ , we consider the relationship between the group  $G$  with the collection  $\mathcal{F}$  of its finite subgroups and the quotient group  $G/K$  with the collection  $\mathcal{F}/K$  of its finite subgroups. For  $G$  we have the ring  $\mathcal{O}_{\mathcal{F}}$  and for  $G/K$  we have the ring

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K})$$

where we have identified finite subgroups of  $G/K$  with their inverse images in  $G$ , i.e., with subgroups  $\tilde{K}$  of  $G$  having identity component  $K$ . Combining the inflation maps associated to passing to quotients by  $K$  for individual groups, there is an inflation map

$$\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}.$$

The second condition is that the object should be *extended*, in the sense that for each connected subgroup  $K$  there is a specified isomorphism

$$M(K) \cong \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

for some  $\mathcal{O}_{\mathcal{F}/K}$ -module  $\phi^K M$ , which is a given part of the structure. These identifications should be compatible when we have inclusions of connected subgroups. If we choose a subgroup  $L$  and then the modules  $\phi^K M$  for  $K \supseteq L$  fit together to make an object of  $\mathcal{A}(G/L)$ .

**4.B. Connection with topology.** The connection between  $G$ -spectra and  $\mathcal{A}(G)$  is given by a homotopy functor

$$\pi_*^{\mathcal{A}} : G\text{-spectra} \longrightarrow \mathcal{A}(G)$$

with the exactness properties of a homology theory. It is rather easy to write down the value of the functor as a diagram of abelian groups.

**Definition 4.1.** For a  $G$ -spectrum  $X$  we define  $\pi_*^{\mathcal{A}}(X)$  on  $K$  by

$$\pi_*^{\mathcal{A}}(X)(K) = \pi_*^G(DEF_+ \wedge S^{\infty W(K)} \wedge X).$$

Here  $EF_+$  is the universal space for the family  $\mathcal{F}$  of finite subgroups with a disjoint basepoint added and  $DEF_+ = F(EF_+, S^0)$  is its functional dual (the function  $G$ -spectrum of maps from  $EF_+$  to  $S^0$ ). The  $G$ -space  $S^{\infty W(K)}$  is defined by

$$S^{\infty W(K)} = \lim_{\rightarrow V^K=0} S^V,$$

when  $K \subseteq H$ , so there is a map  $S^{\infty W(K)} \longrightarrow S^{\infty W(H)}$  inducing the map  $\pi_*^{\mathcal{A}}(X)(K) \longrightarrow \pi_*^{\mathcal{A}}(X)(H)$ .  $\square$

The definition of  $\pi_*^{\mathcal{A}}(X)$  shows that quasi-coherence for  $\pi_*^{\mathcal{A}}(X)$  is just a matter of understanding Euler classes. The extendedness of  $\pi_*^{\mathcal{A}}(X)$  is a little more subtle, and will play a significant role later. We take

$$\phi^K \pi_*^{\mathcal{A}}(X) = \pi_*^{G/K}(DEF/K_+ \wedge \Phi^K(X)),$$

where  $\Phi^K$  is the geometric fixed point functor, and the extendedness follows from properties of the geometric fixed point functor.

To see that  $\pi_*^{\mathcal{A}}(X)$  is a module over  $\mathcal{O}$ , the key is to understand  $S^0$ .

**Theorem 4.2.** [11, 1.5] *The image of  $S^0$  in  $\mathcal{A}(G)$  is the structure functor:*

$$\tilde{\mathcal{O}}_{\mathcal{F}} = \pi_*^{\mathcal{A}}(S^0),$$

*with the canonical structure as an extended module.*

Some additional work confirms that  $\pi_*^{\mathcal{A}}$  has the appropriate behaviour.

**Corollary 4.3.** [11, 1.6] *The functor  $\pi_*^{\mathcal{A}}$  takes values in the abelian category  $\mathcal{A}(G)$ .*

**4.C. Geometric fixed points, geometric isotropy and basic cells.** Since the model was built on the idea that geometric fixed points are fundamental it is easy to read off the algebraic counterparts.

**Remark 4.4.** The counterpart of the geometric  $K$ -fixed points in  $\mathcal{A}(G)$  is obtained by restricting attention to subgroups over  $K$ . More precisely, if  $H \supseteq K$  then

$$\phi^{H/K}(\Phi^K X) = \phi^H X,$$

or equivalently

$$X(H) = \mathcal{E}_H^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \mathcal{E}_{H/K}^{-1} \mathcal{O}_{\mathcal{F}/K} \otimes_{\mathcal{O}_{\mathcal{F}/H}} \phi^H X = \mathcal{E}_H^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} (\Phi^K X)(H/K).$$

**Remark 4.5.** The geometric fixed points of an object  $X$  of  $d\mathcal{A}(G)$  are determined in terms of the algebraic model by

$$\mathcal{I}_g(X) = \{K \mid H_* X(K) \neq 0\}.$$

**Remark 4.6.** If  $\tilde{K}$  is a subgroup with identity component  $K$  then the algebraic model of the basic cell  $\sigma_{\tilde{K}}$  is  $f_K(\mathbb{Q}_{\tilde{K}})$ . In other words, it is determined by its value at the identity component of  $K$ , and at that point it is extended from the  $\mathcal{O}_{\mathcal{F}/K}$ -module  $\mathbb{Q}$  pulled back from  $H^*(BG/\tilde{K})$ . In fact, the value is  $\mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{H^*(BG/\tilde{K})} \mathbb{Q}$ .

Its value at  $L$  is zero unless  $L \subseteq K$ , where the value is  $\mathcal{E}_L^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{H^*(BG/\tilde{K})} \mathbb{Q}$ .

**4.D. The Adams spectral sequence.** The homology theory  $\pi_*^{\mathcal{A}}$  may be used as the basis of an Adams spectral sequence for calculating maps between rational  $G$ -spectra. The main theorem of [11] is as follows.

**Theorem 4.7.** ([11, 9.1]) *For any rational  $G$ -spectra  $X$  and  $Y$  there is a natural Adams spectral sequence*

$$\mathrm{Ext}_{\mathcal{A}(G)}^{*,*}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \Rightarrow [X, Y]_*^G.$$

*It is a finite spectral sequence concentrated in rows 0 to  $r$  (the rank of  $G$ ) and strongly convergent for all  $X$  and  $Y$ .*  $\square$

## 5. GEOMETRIC ISOTROPY OF FINITE SPECTRA

This section is the core of the entire paper. We show that thick tensor ideals are completely determined by the basic cells they contain. In one direction, we show that if  $K$  is maximal in the geometric isotropy of  $X$ , then  $\sigma_K$  is in  $\text{thick}_\otimes(X)$ , and in the other that basic cells construct everything that their geometric isotropy suggests. This comes down to a detailed calculational understanding of maps out of basic cells.

The following finiteness theorem is fundamental.

**Lemma 5.1.** *If  $X$  is a finite spectrum then  $\mathcal{I}_g(X)$  has finitely many maximal elements.*

**Proof:** By Lemma 3.8, we may suppose that  $X \simeq X'$  where  $X'$  is constructed from basic cells  $\sigma_K$  with  $K$  from  $\Lambda_{ct}\mathcal{I}_g(X)$ . Since  $X \simeq X'$ , the identity factors as  $X \rightarrow X' \rightarrow X$ , and since  $X$  is small, we see  $X$  is a retract of a finite subcomplex of  $X'$ . In other words,  $X$  is a retract of a finite complex using cells  $\sigma_{K_1}, \dots, \sigma_{K_N}$  with  $K_i \in \Lambda_{ct}\mathcal{I}_g(X)$ . and hence

$$\text{supp}(X) \subseteq \bigcup_i \Lambda_{ct}(K_i).$$

Forming the dual tower, we see that  $X$  may be converted to a point using the basic cells  $\sigma_{K_1}, \dots, \sigma_{K_N}$ . Elements  $K$  of  $\mathcal{I}_g(X)$  can only be removed if one of the cells is  $\sigma_H$  with  $K \in \Lambda_{ct}H$ , and hence the maximal elements of  $\mathcal{I}_g(X)$  must be amongst the  $K_i$ .  $\square$

**Lemma 5.2.** *If  $A$  is a set of primes with finitely many maximal elements there is a finite spectrum with  $\mathcal{I}_g(X) = A$ .*

**Proof:** We simply enumerate the maximal elements  $K_1, \dots, K_N$  and take  $X = \sigma_{K_1} \vee \dots \vee \sigma_{K_N}$ . The result follows from Lemma 3.3.  $\square$

**Lemma 5.3.** *If  $\mathcal{I}_g(X) \subseteq \Lambda_{ct}(K_1) \cup \dots \cup \Lambda_{ct}(K_N)$  then  $X \in \text{thick}_\otimes(\sigma_{K_1}, \dots, \sigma_{K_N})$ .*

**Proof:** It suffices to show we can kill top isotropy. Indeed, if  $K$  is maximal in  $\mathcal{I}_g(X)$  we must show that the isotropy can be killed by attaching finitely many cells  $\sigma_K$ . In fact, since  $K$  is maximal,  $T = H_{G/K}^*(\Phi^K DX)$  is torsion, and since it is also finitely generated, this is a finite dimensional vector space. We show that this dimension can be reduced by adding a basic cell  $\sigma_K$ .

Indeed, we may define  $X'$  by the cofibre sequence

$$X \xrightarrow{\alpha} f_K(T) \rightarrow X',$$

where  $\alpha$  is the identity at  $K$ . Accordingly,  $K$  is not in  $\mathcal{I}_g(X')$ . Picking  $t \in T$  of lowest degree  $n$  we see there is a map  $\alpha : \sigma_K^n \rightarrow f_K(T)$  hitting  $t$ . Indeed, we consider the Adams spectral sequence for calculating  $[\sigma_K, f_K(T)]_*^G$ . Its  $E_2$ -term is easy to understand, since

$$\text{Ext}_{\mathcal{A}(G)}^{*,*}(\pi_*^A(\sigma_K), \pi_*^A(f_K(T))) = \text{Ext}_{H^*(BG/K)}^{*,*}(\pi_*^A(\sigma_K)(K), T) = \text{Ext}_{H^*(BG/K)}^{*,*}(\mathbb{Q}, T),$$

and we see by degree that  $t \in \text{Hom}_{H^*(BG/K)}^*(\mathbb{Q}, T)$  is an infinite cycle.

**Lemma 5.4.** *Since  $K \notin \mathcal{I}_g(X')$  the module  $[\sigma_K, X']$  is annihilated by  $\mathcal{E}_K$*

**Proof:** First note that by the nature of injectives, the property that  $X'(K) = 0$  is inherited by its injective resolution. It therefore suffices to show  $\text{Hom}(\sigma_K, Y')$  is torsion when  $Y'(K) = 0$ . Next, note that a map from  $\sigma_K$  is determined by its value at 1. Now consider the square

$$\begin{array}{ccc} \sigma_K(K) & \longrightarrow & Y'(K) = 0 \\ \uparrow & & \uparrow \\ \sigma_K(1) & \longrightarrow & Y'(1) \end{array}$$

It follows that the image of each element of  $x \in \sigma_K(1)$  is annihilated by an Euler class  $e(W_x)$  with  $W_x^K = 0$ . Since  $\sigma_K(L)$  is finitely generated, there is a single representation  $W$  independent of  $x$  with  $W^K = 0$  so that  $e(W)$  annihilates the whole image of  $\sigma_K(1)$   $\square$

By the lemma, we may find a  $K$ -essential representation  $W$  so that the right hand diagonal composite

$$\begin{array}{ccccc} & & S^{-W} \wedge \sigma_K^n & & \\ & \swarrow \widetilde{e(W)\alpha} & \downarrow e(W) & \searrow 0 & \\ & X & \sigma_K^n & & X' \\ & \searrow & \downarrow \alpha & \nearrow & \\ X & \longrightarrow & f_K(T) & \longrightarrow & X' \end{array}$$

is zero, and so  $\widetilde{e(W)\alpha}$  lifts to a map  $\widetilde{e(W)\alpha}$  to  $X$ .

We note that  $\alpha$  is nonzero as a map  $\mathbb{Q} \longrightarrow T$  (i.e., more precisely, this is the map  $\phi^K \alpha$ ), and hence the same is true of  $\widetilde{e(W)\alpha}$ . Accordingly, if we take  $X_1 = \text{cofibre}(\widetilde{e(W)\alpha})$  the vector space  $H_{G/K}^*(\Phi^K DX_1)$  is of smaller dimension. After adding finitely many cells we reach  $X_N$  with  $H_{G/K}^*(\Phi^K DX_1) = 0$ , so that  $K \notin \mathcal{I}_g(X_N)$ , and we have a cofibre sequence

$$X \longrightarrow X_N \longrightarrow C$$

where  $C$  is built from suspensions of copies of  $\sigma_K$ .

By the triangle property of geometric isotropy,

$$\mathcal{I}_g(X_N) \subseteq \mathcal{I}_g(X) \cup \Lambda_{ct}(K) = \mathcal{I}_g(X).$$

In finitely many steps we may use the above to remove all top dimensional geometric isotropy. Repeating this finitely many times we are left with empty geometric isotropy and hence a contractible object by the Geometric Fixed Point Whitehead Theorem.  $\square$

**Theorem 5.5.** *If  $K$  is maximal in  $\mathcal{I}_g(X)$  then  $\sigma_K \in \text{thick}_{\otimes}(X)$*

**Proof:** For the particular group  $G$  we will argue by induction on the dimension of the subgroup  $K$  maximal in  $\mathcal{I}_g(X)$ . If  $K$  is 0 dimensional then  $f_1(X(K))$  is a retract of  $X$ , and  $X(K)$  is a torsion module over  $H^*(BG/K)$ . It is well known that any two finitely generated torsion modules over  $H^*(BG/K)$  generate the same thick subcategory, so  $\mathbb{Q}$  is finitely built by  $X(K)$  and  $\sigma_K$  is finitely built by  $f_1(X(K))$ .

Now suppose  $K$  is of dimension  $d$  and that the result is true for those of smaller dimension. If  $L$  is a proper cotoral subgroup of  $K$  then  $L$  is a maximal subgroup of  $\mathcal{I}_g(X \wedge \sigma_L)$  of smaller

dimension, and hence by induction  $\sigma_L \in \text{thick}_\otimes(X \wedge \sigma_L) \subseteq \text{thick}_\otimes(X)$ . We will use this Inductive Observation several times below. It remains to show that  $\sigma_K$  is in  $\text{thick}_\otimes(X)$ .

As a warm-up, we will first consider the special case  $K = G$  (those fluent with  $\mathcal{A}(G)$  can skip straight to the general case). Now  $X(G) = \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$  for a graded vector space  $V$ , and there is a map  $i : X \rightarrow f_G(V)$  which is the identity at  $G$ . However  $f_G(V)$  is not a finite spectrum. We claim that the image of  $i$  lies inside a wedge of spheres  $WS$ .

Note that since  $X$  is finite,  $V$  is finite dimensional and we may write  $V = \bigoplus_i \Sigma^{n_i} \mathbb{Q}$ . The most obvious finite spectrum  $Y$  with  $\Phi^G Y = V$  is  $\bigvee_i S^{n_i}$ , but typically the image of  $X$  will not be a subspace of this. However, if we suspend this wedge of spheres by any representation  $W$  with  $W^G = 0$  we obtain another candidate. On the other hand, the quotient of  $f_G(V)(1) = \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$  by  $iX(1)$  is  $\mathcal{E}_G$ -torsion, and by finiteness of  $X$  there is a single complex representation  $W$  so that the Euler class  $c(W)$  annihilates it. and the sum is finite since  $X$  (and hence  $\Phi^G X$ ) is finite.

Hence we find

$$i(X) \subseteq S^{-W} \wedge \bigvee_i S^{n_i} =: WS.$$

We may now consider the cofibre sequence

$$X \rightarrow WS \rightarrow Y.$$

Since  $X$  and  $WS$  are small so is  $Y$ . Since  $X \rightarrow WS$  is an isomorphism at  $G$ , the group  $G$  is not in the geometric isotropy of  $Y$ , so  $Y$  is constructed from  $\sigma_L$  with  $L$  proper. By the Induction Observation these basic cells can be built from  $X$ , so  $WS$  is in  $\text{thick}_\otimes(X)$ . Any one of the spheres  $S^{n_i-W}$  is a retract of  $WS$  and a suspension of it is in the thick tensor ideal, so that

$$S^0 \in \text{thick}_\otimes(WS) \subseteq \text{thick}_\otimes(X).$$

Now we turn to general case. First,  $X(K) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} T$  for a torsion  $\mathcal{O}_{\mathcal{F}/K}$ -module  $T$ . By the maximality of  $K$  and the adjointness property of  $f_K$ , there is a map  $i : X \rightarrow f_K(T)$  which is the identity at  $K$ . However  $f_K(T)$  will typically not be a finite spectrum.

We construct a finite complex  $FC'$  from suspensions of copies of  $\sigma_K$  to play the role of  $f_K(T)$ . Indeed, we use the procedure of Lemma 5.3 to construct a map  $f' : FC' \rightarrow f_K(T)$  which is an isomorphism on  $\phi^K$ . The quotient of  $f_K(T)(1) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} T$  by  $iX(K)$  is  $\mathcal{E}_K$ -torsion, so there is a representation  $W$  with  $W^K = 0$  annihilating it. Accordingly we may take  $FC = \Sigma^W FC'$  and obtain a map  $f : FC = \Sigma^W FC' \rightarrow f_K(T)$  which is the identity at  $K$  and with  $\text{im}(i) \subseteq \text{im}(f)$ .

We may now consider the cofibre sequence

$$X \rightarrow FC \rightarrow Y.$$

Since  $X$  and  $FC$  are small so is  $Y$ . Since  $X \rightarrow FC$  is an isomorphism at  $K$ , the group  $K$  is not in the geometric isotropy of  $Y$ , so  $Y$  is constructed from  $\sigma_L$  with  $L$  a proper cotoral subgroup of  $K$ . By the Inductive Observation, these basic cells can be built from  $X$ , so  $FC$  is in  $\text{thick}_\otimes(X)$ .

Finally,  $\sigma_K \in \text{thick}_\otimes(FC)$ . This can be proved in various ways, and we will use Hopkins's method [16]. Suppose the top dimensional basic cells of  $FC$  are of dimension  $n$ . They are all attached to the basic  $(n-1)$ -skeleton, so we can add them in any order. In particular, there is a map  $FC \rightarrow \sigma_K^n$  which is surjective in  $H_*$ . Accordingly the mapping cone  $f : \sigma_K^n \rightarrow R$  is zero in homology and hence tensor nilpotent. Now  $\Sigma FC = C(f)$  so  $C(f) \otimes \sigma_K \in \text{thick}_\otimes(FC)$ ,

and hence  $C(f^n) \otimes k \in \text{thick}_\otimes(FC)$ . If  $f^n \simeq 0$  then  $\sigma_K$  is a retract of  $C(f^n)$ .  $\square$

**Corollary 5.6.** *If  $X$  and  $Y$  are finite spectra with  $\mathcal{I}_g(X) = \mathcal{I}_g(Y)$  then  $\text{thick}_\otimes(X) = \text{thick}_\otimes(Y)$ .*  $\square$

**Proof:** If  $K_1, \dots, K_N$  are the maximal elements of  $\mathcal{I}_g(X)$  then  $\sigma_{K_i} \in \text{thick}_\otimes(X)$  by Theorem 5.5. Hence  $\text{thick}_\otimes(X) \supseteq \bigcup_i \Lambda_{ct}(K_i)$  by Lemma 5.3. The reverse inclusion is obvious.  $\square$

Some may prefer the following reformulation. A thick tensor ideal is called *finitely generated* if it is generated by a finite number of small spectra (or equivalently, by just one).

**Corollary 5.7.** *The finitely generated thick tensor ideals of finite rational  $G$ -spectra are precisely the fibres of geometric isotropy*

$$\mathcal{I}_g : \text{finite-rational-}G\text{-spectra} \longrightarrow \mathcal{P}(\text{Sub}(G));$$

*the image consists of collections of subgroups with finitely many cotorally maximal elements that are closed under cotoral specialization.*

**Remark 5.8.** The finite generation hypothesis is necessary, since for example the collection of finite  $\mathcal{F}$ -spectra is a thick tensor ideal but its geometric isotropy has infinitely many maximal elements.

## 6. PRIMES

It is interesting to reformulate our classification in terms of Balmer prime spectra. In fact, the calculation of Balmer spectra was the initial objective of this project, but it only made progress when formulated in terms of geometric isotropy and thick tensor ideals. In effect this shows the power of considering supports.

**6.A. Background.** We recall the basic definitions from [1].

**Definition 6.1.** A *prime ideal* in a tensor triangulated category is a thick proper tensor ideal  $\wp$  with the property that  $ab \in \wp$  implies that  $a$  or  $b$  is in  $\wp$ .

The *Balmer spectrum* of a tensor-triangulated category  $\mathbb{C}$  is  $\text{Spc}(\mathbb{C}) = \{\wp \mid \wp \text{ is prime}\}$ . The Zariski topology on  $\text{Spc}(\mathbb{C})$  is the one generated by the closed sets  $\text{supp}(X) = \{\wp \mid X \notin \wp\}$  as  $X$  runs through objects of  $\mathbb{C}$ .

**Theorem 6.2.** *If  $G$  is a torus then  $\text{Spc}((G\text{-spec}/\mathbb{Q})^c)$  is the partially ordered set  $\text{Sub}_a(G)$  whose elements are closed subgroups with  $L \leq K$  if  $L \subseteq K$  with  $K/L$  a torus.*

It is rather easy to find  $\text{Sub}_a(G)$  inside  $\text{Spc}((G\text{-spec}/\mathbb{Q})^c)$ . The main work will be to show that we have found all the primes.

To start with we consider

$$\wp_K = \{X \mid \Phi^K X \simeq_1 0\}.$$

To see this is prime we note that 0 is a prime in rational spectra (since it is equivalent to chain complexes of  $\mathbb{Q}$ -modules) and

$$\wp_K = (\Phi^K)^*((0)) \text{ where } \Phi^K : G\text{-spectra} \longrightarrow \text{spectra}.$$

**Lemma 6.3.** *For any finite spectrum  $X$ , the support in the sense of Balmer for this set of primes coincides with the geometric isotropy:*

$$\text{supp}(X) = \{H \mid X \notin \wp_H\} = \{H \mid \Phi^H X \not\simeq 0\} = \mathcal{I}_g(X). \quad \square$$

We will in due course show that the  $\wp_K$  give all primes, so that this will be the full Balmer support.

**Lemma 6.4.** *The intersection of all these primes is trivial*

$$\bigcap_K \wp_K = 0.$$

**Proof:** This says that any spectrum with trivial geometric fixed points for all subgroups is contractible, which is the Geometric Fixed Point Whitehead Theorem.  $\square$

We should note that these tensor ideals are not finitely generated.

**Lemma 6.5.** *The geometric isotropy of  $\wp_K$  is*

$$\mathcal{I}_g(\wp_K) = \{H \mid K \not\leq H\} = \bigcup_{K \not\leq H} \Lambda_{ct}(H).$$

Thus  $\sigma_H \in \wp_K$  whenever  $K \not\leq H$ .  $\square$

**6.B. The lattice of subgroup primes.** The Localization Theorem shows that containment of subgroup primes precisely corresponds to cotoral inclusion.

**Lemma 6.6.**  *$\wp_L \subseteq \wp_K$  if and only if  $L \subseteq K$  with  $K/L$  a torus.*

**Proof:** First suppose  $L$  is cotoral in  $K$ . We need to show that  $\Phi^L X \simeq 0$  (non-equivariantly) implies  $\Phi^K X \simeq 0$  (non-equivariantly). This follows from the Localization Theorem. In more detail, we may as well consider the  $K/L$ -spectrum  $Y = \Phi^L X$ . The hypothesis is that  $Y$  is nonequivariantly contractible (ie that  $EK/L_+ \wedge Y \simeq 0$ ) and the conclusion is that  $\Phi^{K/L} Y$  is nonequivariantly contractible. However the localization theorem states that for finite  $K/L$ -complexes  $Z$  the map

$$H_{K/L}^*(Z) \longrightarrow H_{K/L}^*(\Phi^{K/L} Z) = H^*(BK/L) \otimes H^*(\Phi^{K/L} Z)$$

becomes an isomorphism when  $\mathcal{E}_{K/L}$  is inverted. Taking  $Z = DY$  gives the required result since  $\mathcal{E}_{K/L}^{-1} H^*(BK/L)$  is nonzero.

If  $L$  is not a subgroup of  $K$  then  $G/K_+$  has trivial  $L$ -fixed points and non-trivial  $K$ -fixed points so  $\wp_L \not\subseteq \wp_K$ . Finally, suppose  $L$  is a subgroup of  $K$ , but not cotoral. We may consider the  $K$ -spectrum  $\sigma_K$ ; by Lemma 3.3  $\Phi^K \sigma_K \not\simeq 0$  but  $\Phi^L \sigma_K \simeq 0$  so  $\wp_L \not\subseteq \wp_K$ .  $\square$

## 7. THE SUBGROUP PRIMES EXHAUST THE PRIMES

Finally, in this section we deduce the structure of the Balmer spectrum from our classification of thick tensor ideals. The main point is to show that all primes are of the form  $\wp_H$  for some closed subgroup  $H$ .



**7.A. Thick tensor ideals and primes.** The relationship between thick tensor ideals and primes is easily deduced from the classification of thick tensor ideals Corollary 5.6.

**Corollary 7.1.** *The thick subcategory generated by a finite rational spectrum  $X$  is an intersection of primes:*

$$\text{thick}_{\otimes}(X) = \bigcap_{K \notin \mathcal{I}_g(X)} \wp_K. \quad \square$$

**7.B. Exhaustion.** We finally want to prove that we have found all the primes.

**Theorem 7.2.** *Every prime  $\wp$  of  $(G\text{-spec}/\mathbb{Q})^c$  is  $\wp_K$  for some closed subgroup  $K$ .*

This essentially follows from Corollary 5.7. We first observe that it suffices to show that an arbitrary prime is an intersection of those of the form  $\wp_L$ .

**Lemma 7.3.** *If  $\wp$  is a prime and  $\wp = \bigcap_{L \in A} \wp_L$  then  $A$  has a unique minimal element  $L$  and  $\wp = \wp_L$ .*

**Proof:** If not we can choose a subgroup  $L$  in  $A$  which is not redundant. Thus

$$\wp = \wp_L \cap \bigcap_{K \in A \setminus \{L\}} \wp_K,$$

and since  $\wp_L$  is not redundant, we may choose  $X_L \in \wp_L \setminus \wp$  and  $Y_L \in \bigcap_{K \in A \setminus \{L\}} \wp_K \setminus \wp$ . This contradicts the fact that  $\wp$  is prime since  $X_L \wedge Y_L \in \wp$  but  $X_L \notin \wp$  and  $Y_L \notin \wp$ .  $\square$

**Proof of 7.2:** Write  $\mathcal{I}_g(\wp) = \bigcup_{X \in \wp} \mathcal{I}_g(X)$ . This is a countable set, so we can list its elements  $K_1, K_2, \dots$ . If  $K \in \mathcal{I}_g(\wp)$  then  $\Lambda_{ct}(K) = \mathcal{I}_g(\sigma_K) \subseteq \mathcal{I}_g(\wp)$  and hence  $\sigma_K \in \wp$ .

Hence

$$\wp = \bigcup_i \text{thick}(\sigma_{K_i})$$

Since

$$\begin{aligned} \text{thick}(\sigma_K) &= \bigcap_{L \not\leq K} \wp_L \\ \wp &= \bigcap_{L \notin \bigcup_i \Lambda_{ct}(K_i)} \wp_L \end{aligned}$$

By Lemma 7.3 if there is more than one minimal prime in this list  $\wp$  is not prime.  $\square$

**7.C. The Zariski topology.** We observe that the topology is entirely generated by containments. Noting that Balmer primes reverse the inclusions of commutative algebra primes, closed sets are closed under passage to smaller primes.

**Lemma 7.4.** *The Zariski topology on  $\text{Spc}((G\text{-spec}/\mathbb{Q})^c)$  is generated by the closed sets  $\Lambda(\wp_H)$ .*

**Proof:** Since  $\mathcal{I}_g(\sigma_K) = \Lambda_{ct}(K)$ , we see  $\Lambda(\wp_H) = \{\wp_K \mid K \leq H\} = \text{supp}(\sigma_K)$ , and the specified subsets are closed.

Again, for a finite spectrum  $X$ , Lemma 5.1 shows  $\text{supp}(X)$  has finitely many maximal elements  $K_1, \dots, K_N$ , so that  $\text{supp}(X) = \text{supp}(\sigma_{K_1}) \cup \dots \cup \text{supp}(\sigma_{K_N})$ .  $\square$

## 8. SEMIFREE $\mathbb{T}$ -SPECTRA

The point of this section is to show that it is much harder to classify thick subcategories than thick tensor-ideals. It will suffice to look at semifree  $\mathbb{T}$ -spectra, i.e., those  $\mathbb{T}$ -spectra with  $\mathcal{I}_g(X) \subseteq \{1, \mathbb{T}\}$ . The model for these [9] is sufficiently simple that we may be explicit.

**8.A. The model of semifree  $\mathbb{T}$ -spectra.** The model  $\mathcal{A}_{sf}(T)$  of semifree  $T$ -spectra can be obtained from the model  $\mathcal{A}(T)$  of all  $T$ -spectra by restriction, but it is easier to repeat the construction from scratch. In fact  $\mathcal{A}_{sf}(T)$  is the abelian category of objects  $\beta : N \rightarrow \mathbb{Q}[c, c^{-1}] \otimes V$  where  $N$  is a  $\mathbb{Q}[c]$ -module,  $V$  is a graded  $\mathbb{Q}$ -vector space and  $\beta$  is the  $\mathbb{Q}[c]$  map inverting  $c$ . In effect, we have the  $\mathbb{Q}[c]$ -module  $N$ , together with a chosen ‘basis’  $V$ . Morphisms are commutative squares

$$\begin{array}{ccc} M & \xrightarrow{\theta} & N \\ \downarrow & & \downarrow \\ \mathbb{Q}[c, c^{-1}] \otimes U & \xrightarrow{1 \otimes \phi} & \mathbb{Q}[c, c^{-1}] \otimes V \end{array}$$

The category  $\mathcal{A}_{sf}(T)$  is of injective dimension 1, and the ring  $\mathbb{Q}[c]$  is evenly graded, so every object of  $d\mathcal{A}_{sf}(T)$  is formal, and we will identify semifree  $T$ -spectra  $X$  (or objects of  $d\mathcal{A}_{sf}(T)$ ) with their image  $\pi_*^{\mathcal{A}}(X)$  in the abelian category  $\mathcal{A}_{sf}(T)$ .

The fact we are talking about *ideals* is essential for Corollary 5.7. If we consider semifree  $G$ -spectra when  $G$  is the circle then there are just two thick tensor ideals of finite spectra

- free spectra (with geometric isotropy 1, generated by  $G_+$ )
- all spectra (with geometric isotropy  $\{1, G\}$ , generated by  $S^0$ ).

On the other hand, the thick subcategory generated by  $S^0$  (without the ideal property) does not contain  $G_+$ , and we will make it explicit. The classification of thick subcategories in general seems complicated, and we do not give a complete answer.

**8.B. Wide spheres.** The small objects with  $\beta$  injective are the objects  $X = (\beta : N \rightarrow \mathbb{Q}[c, c^{-1}] \otimes V)$  with  $\beta$  injective,  $V$  finite dimensional and  $N$  bounded above; these objects are called *wide spheres* [9].

We note that  $\mathbb{Q}[c, c^{-1}] \otimes V$  is the same in each even degree and the same in each odd degree. We therefore let

$$|V|_0 = \bigoplus_k V_{2k} \text{ and } |V|_1 = \bigoplus_k V_{2k+1}.$$

We will fix isomorphisms

$$|V|_0 \cong (\mathbb{Q}[c, c^{-1}] \otimes V)_0 \text{ and } |V|_1 \cong (\mathbb{Q}[c, c^{-1}] \otimes V)_1,$$

and then use multiplication by powers of  $c$  to identify other graded pieces of  $\mathbb{Q}[c, c^{-1}] \otimes V$  with the appropriate one.

We will want to think of stepping down the degrees in steps of 2, so we take

$$|V|_{\geq 2k} = \bigoplus_{n \geq k} V_{2n} \text{ and } |V|_{\geq 2k+1} = \bigoplus_{n \geq k} V_{2n+1}$$

for the parts of  $V$  above a certain point, but in the same parity.

Similarly, we move  $N_{2k}$  into degree 0 by multiplication by  $c^k$ :

$$\overline{N}_{2k} := c^k N_{2k} \subseteq |V|_0 \text{ and } \overline{N}_{2k+1} := c^k N_{2k+1} \subseteq |V|_1.$$

Having established the framework, we will restrict the discussion to the even part, leaving the reader to make the odd case explicit.

If  $X$  is nonzero in even degrees, since  $X$  is small there is a highest degree  $2a-2$  in which  $N$  is non-zero, and since  $N[1/c] = \mathbb{Q}[c, c^{-1}] \otimes V$  there is highest degree  $2b$  in which  $N$  coincides with  $|V|_0$ . Accordingly, we have a finite filtration

$$0 = \overline{N}_{2a} \subseteq \overline{N}_{2a-2} \subseteq \cdots \subseteq \overline{N}_4 \subseteq \overline{N}_2 \subseteq \cdots \subseteq \overline{N}_{2b} = |V|_0.$$

We wish to consider two increasing filtrations on  $|V|_0$

$$\cdots \subseteq |V|_{\geq 2k+2} \subseteq |V|_{\geq 2k} \subseteq |V|_{\geq 2k-2} \subseteq \cdots \subseteq |V|_0$$

and

$$\cdots \subseteq \overline{N}_{2k+2} \subseteq \overline{N}_{\geq 2k} \subseteq \overline{N}_{\geq 2k-2} \subseteq \cdots \subseteq |V|_0.$$

**8.C. Two conditions on wide spheres.** In crude terms, we will show the thick subcategory generated by  $S^0$  consists of objects so that (a) the dimensions of the spaces in the  $V$ - and  $N$ -filtrations agree and (b) the  $V$  filtration is slower than the  $cN$  filtration. The purpose of this subsection is to introduce the two conditions.

**Condition 8.1.** We say that a wide sphere is *untwisted* if it satisfies the following two conditions

- (i)  $\dim(\overline{N}_i) = \dim(|V|_{\geq i})$  for all  $i$
- (ii)  $V \cap cN = 0$

We will be showing that these characterize the thick subcategory generated by  $S^0$ . We must at least show that the conditions are inherited by retracts, and this verification will lead us to some useful introductory discussion.

**Lemma 8.2.** *Condition 8.1 is closed under passage to retracts.*

**Proof:** It is immediate that Condition 8.1 (ii) is inherited by retracts. We also note that Condition 8.1 (ii) implies one of the inequalities for Condition 8.1 (i):

$$\dim(\overline{N}_i) \geq \dim(|V|_{\geq i}).$$

Now suppose  $X = X' \oplus X''$  and that  $X$  satisfies Condition 8.1. As observed already,  $X'$  and  $X''$  both satisfy the second condition, and hence both satisfy the first condition with  $=$  replaced by  $\geq$ . With lower case letters denoting dimensions of vector spaces (for example  $n_a = \dim(N_a)$ ), this means we have a pair of increasing sequences

$$0 = n'_a, n'_{a-2}, n'_{a-4}, \cdots \text{ and } 0 = v'_{\geq a}, v'_{\geq a-2}, v'_{\geq a-4} \cdots$$

reaching  $v'$  and a pair of increasing sequences

$$0 = n''_a, n''_{a-2}, n''_{a-4}, \cdots \text{ and } 0 = v''_{\geq a}, v''_{\geq a-2}, v''_{\geq a-4} \cdots$$

reaching  $v''$ . Since  $X$  satisfies Condition 8.1(i), the sum of the first pair and the second pair give two equal sequences (i.e., the sequence  $n'_i + n''_i = n_i$  and the sequence  $v'_{\geq i} + v''_{\geq i} = v_{\geq i}$  are equal). Thus if one pair deviates from equality in the positive direction, the other deviates in the negative direction. Since Condition 8.1(ii) shows there is no negative deviation, we must have equality for both pairs throughout.  $\square$

It is useful to be able to consider the changes of dimension and form the generating function. In fact to any wide sphere, we may associate to it two Laurent polynomials

- The geometric  $T$ -fixed point polynomial

$$p_T(t) = \sum_i \dim_{\mathbb{Q}}(V_i) t^i$$

- The 1-Borel jump polynomial

$$p_1(t) = \sum_i \dim_{\mathbb{Q}}(N_i/cN_{i+2}) t^i$$

Condition 8.1(i) is then equivalent to the condition

$$p_T(t) = p_1(t).$$

**Remark 8.3.** We note that Condition 8.1(i) is not closed under passage to retracts. Indeed,  $S^z \vee S^{2-z}$  satisfies the first condition with  $p_1(t) = p_T(t) = t^2 + 1$ . However  $S^z$  (with  $p_T(t) = 1$  and  $p_1(T) = t^2$ ) and  $S^{2-z}$  (with  $p_T(t) = t^2$  and  $p_1(T) = 1$ ) do not.

**8.D. Attaching a  $T$ -fixed sphere.** To start with,  $S^0 = (\mathbb{Q}[c] \rightarrow \mathbb{Q}[c, c^{-1}] \otimes \mathbb{Q})$  and then direct sums of these model wedges of  $T$ -fixed spheres with  $N = \mathbb{Q}[c] \otimes V$ , and of course it is easy to see that Condition 8.1 holds for these.

However  $N$  does not always sit so simply inside  $\mathbb{Q}[c, c^{-1}] \otimes V$  for the objects built from  $S^0$ . We may see this in a simple example.

**Example 8.4.** Up to equivalence there are precisely three wide spheres with  $p_1(t) = p_T(t) = 1 + t^2$ . Evidently in all cases  $V = \mathbb{Q} \oplus \Sigma^2 \mathbb{Q}$ ,  $\overline{N}_{2k} = 0$  for  $k \geq 4$  and  $\overline{N}_{2k} = |V|$  for  $k \leq 0$ . The only question is how the 1-dimensional space  $\overline{N}_2$  sits inside  $|V| = V_0 \oplus V_2$ . The three cases are  $N_2 = V_0$  (which is  $S^z \vee S^{2-z}$ ),  $N_2 = V_2$  (which is  $S^0 \vee S^2$ ), and the third case (giving just one isomorphism type) in which  $N_2$  is a 1-dimensional subspace not equal to  $V_0$  or  $V_1$ .

We note that the third example is the mapping cone  $M_f$  for any non-trivial map  $f : S^1 \rightarrow S^0$  (in the semifree category, there is only one up to multiplication by a non-zero scalar). In this case up to isomorphism,  $N_2$  is generated by  $c^{-1} \otimes \iota_0 + c^0 \otimes \iota_2$ .

We observe then that the second and third of these three are in  $\text{thick}(S^0)$ , and we see that the first does not satisfy Condition 8.1 (ii).

**Lemma 8.5.** *Given a cofibre sequence,*

$$S^n \xrightarrow{f} X \rightarrow Y,$$

*if  $X$  is a wide sphere then so is  $Y$  and if  $X$  in addition satisfies Condition 8.1 then so does  $Y$ .*

**Proof:** Suppose first that  $X$  is entirely in one parity. Without loss of generality, we may suppose  $X$  is in even degrees.

If  $n$  is odd then  $\pi_*^{\mathcal{A}}(S^n)$  is purely odd and we have a short exact sequence

$$0 \rightarrow \pi_*^{\mathcal{A}}(X) \rightarrow \pi_*^{\mathcal{A}}(Y) \rightarrow \pi_*^{\mathcal{A}}(S^{n+1}) \rightarrow 0.$$

It follows that  $Y$  is a wide sphere (i.e., the basing map is injective). The condition on dimensions is immediate, since this is split as vector spaces. For the second condition, we

know that any element  $(v, \lambda v) \in V_Y \cap cN_Y$  with  $v \in V_X$  must have  $\lambda \neq 0$  since  $X$  satisfies the condition. However  $\lambda v \notin c\mathbb{Q}[c]$ . Altogether,  $Y$  satisfies Condition 8.1.

Alternatively, suppose  $n$  is even. To calculate  $[S^n, X]_*^T$  we take an injective resolution of  $X$ . We argue that this takes the form

$$0 \longrightarrow X \longrightarrow e(V) \longrightarrow f(\Sigma^2 V \otimes k[c]^\vee) \longrightarrow 0.$$

To start with, since  $X$  is a wide sphere,  $X$  embeds in  $e(V)$ . The cokernel is zero at  $T$  and hence of the form  $f(T)$  for some torsion  $\mathbb{Q}[c]$ -module  $T$ . At 1 the cokernel is  $(\mathbb{Q}[c, c^{-1}] \otimes V)/N$ ; since this is divisible it is a sum of copies of  $\mathbb{Q}[c]^\vee$ . Finally, in view of Condition 8.1 (i)  $T = \Sigma^2 V \otimes \mathbb{Q}[c]^\vee$  as claimed.

Now we may use the Adams spectral sequence to see that  $[S^n, X]_0^T = V_n$ . If the original map  $f$  is trivial, then  $\pi_*^A(Y) = \pi_*^A(X) \oplus \pi_*^A(S^{n+1})$ , and the result is again clear. Otherwise we have a diagram

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ & & \\ \Sigma^n \mathbb{Q}[c] & \xrightarrow{\theta} & N \\ \downarrow & & \downarrow \\ \mathbb{Q}[c, c^{-1}] \otimes \Sigma^n \mathbb{Q} & \xrightarrow{1 \otimes \phi} & \mathbb{Q}[c, c^{-1}] \otimes V \end{array}$$

This shows that since  $\phi$  is mono then also  $\theta$  is mono and furthermore, by Condition 8.1(ii),  $\theta$  is the inclusion of a summand. It follows that the map  $f$  is split. Indeed, splittings of  $\phi$  are given by codimension 1 free summands  $N'$  of  $N$ . This automatically has  $\overline{N}'$  of codimension 1 in  $|V|$ . We need only choose  $N'$  so that  $\overline{N}'$  avoids  $\theta(\Sigma^n \mathbb{Q})$ . This gives a compatible splitting of  $\phi$ . It follows that in fact  $Y$  is a retract of  $X$ , and the result follows from Lemma 8.2.

Finally if  $X$  has components in both even and odd degrees, then  $X \simeq X_{ev} \vee X_{od}$  and we may argue as follows. Without loss of generality we suppose  $n$  is even. If  $f$  maps purely into  $X_{ev}$  or purely into  $X_{od}$  the other factor is irrelevant and the above argument deals with this case. Otherwise  $f$  has components mapping into both  $X_{ev}$  and  $X_{od}$ . The above argument shows that  $\pi_*^A(Y)$  is a retract in even degrees and it is unaltered in odd degrees.  $\square$

**8.E. Spectra built from  $T$ -fixed spheres.** We have now done the main work and can identify the thick subcategory generated by  $S^0$ .

**Corollary 8.6.** *The thick subcategory generated by  $S^0$  consists of wide spheres satisfying Condition 8.1.*

**Proof:** First, we observe that the thick subcategory  $\text{thick}(S^0)$  can be constructed by alternating the attachment of  $T$ -fixed spheres and taking retracts; the fact that any element of  $\text{thick}(S^0)$  is a wide sphere satisfying Condition 8.1 then follows from Lemmas 8.2 and 8.5. The point is that we must show that if we construct  $Z$  using a cofibre sequence  $X \longrightarrow Y \longrightarrow Z$  with  $X, Y$  in the thick subcategory then  $Z$  may be constructed from  $X$  by using the two processes. Formally, we are applying induction on the number of cells, so we may suppose  $Y$  is constructed from the two processes. If  $X$  is formed by attaching spheres, we may form  $Z$  from  $Y$  by attaching the corresponding spheres. If  $X$  is a retract of  $X'$  formed from spheres

then  $f : X \rightarrow Y$  extends over  $X' = X \vee X''$  by using 0 on the second factor and then  $Z$  is a retract of  $Z'$ .

Now we show that any wide sphere satisfying Condition 8.1 is in the thick subcategory generated by  $S^0$ . We argue by induction on the dimension of  $|V|$ . The result is obvious if  $V = 0$ . Suppose that  $X$  is a wide sphere satisfying the given condition and that the result is proved when the geometric  $T$ -fixed points have lower dimension.

Note that if  $t^n$  is the smallest degree in which  $p_T(t)$  is non-zero we may choose a vector  $v \in V_n \setminus \overline{N}_{n+2}$ . Accordingly  $X$  has a direct summand  $\mathbb{Q}[c] \otimes v \rightarrow \mathbb{Q}[c, c^{-1}] \otimes v$ , which corresponds to a map  $S^n \rightarrow X$ . Since  $v \notin \overline{N}_{n+2}$ , the quotient  $Y$  again has injective basing map and obviously satisfies the polynomial condition. Since  $n$  is the smallest degree in which  $V$  is non-zero,  $v$ , the direct summand  $\mathbb{Q} \cdot v$  may be removed from  $|V|$  without affecting the filtration condition. By induction  $Y \in \text{thick}(S^0)$ , and hence  $X \in \text{thick}(S^0)$  as required.  $\square$

**8.F. Spectra built from representation spheres.** Since smashing with any sphere  $S^{kz}$  is invertible, this allows us to deduce the thick subcategory generated by any sphere.

**Corollary 8.7.** *The thick subcategory generated by  $S^{kz}$  consists of wide spheres which are  $k$ -twisted in the sense that*

- (i)  $p_1(t) = t^{-2k} p_T(t)$ .
- (ii)  $V \cap c^{k+1} N = 0$

**Proof:** This is immediate from Corollary 8.6 and the observations

$$V(S^{kz} \wedge X) = V(X) \text{ and } N(S^{kz} \wedge X) = c^{-k} N(X).$$

$\square$

## 9. BEYOND TORI

It helps give a proper perspective to describe the Balmer spectra of other low rank groups. The answer follows easily from the work on the circle group.

**9.A. Finite groups.** For a finite group, the category of rational  $G$ -spectra is equivalent to the product  $\prod_{(H)} \mathbb{Q}[W_G(H)]\text{-mod}$ , where the product is over conjugacy classes of subgroups,

$$\text{Spc}(G\text{-spectra}) = \{\wp_H \mid (H) \in \text{Sub}(G)/G\}.$$

There are no containments between the subgroups and the topological space is discrete.

**9.B. The circle group  $SO(2)$ .** This is the special case of Theorem 7.2 for the torus of rank 1, but we will use it below for  $O(2)$  and  $SO(3)$  so it is worth making it explicit.

The closed subgroups of  $SO(2)$  are the finite cyclic subgroups  $C_n$  and  $SO(2)$  itself. Each of the finite subgroups is cotoral in the whole group. Accordingly, by Theorem 7.2 we have

$$\text{Spc}(\mathcal{A}(SO(2))) = \{\wp_{C_n} \mid n \geq 1\} \cup \{\wp_{SO(2)}\} =$$

$\uparrow$   
 $C_1$

$\nwarrow$   
 $C_2$

$\nwarrow$   
 $C_3$

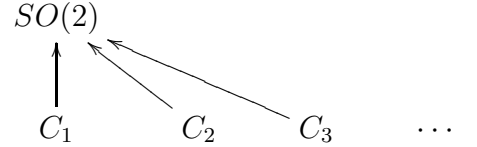
$\dots$

with  $\wp_{C_n} \subset \wp_{SO(2)}$ .

9.C. **The orthogonal group  $O(2)$ .** The homotopy category of  $O(2)$ -spectra splits into two parts.

- The cyclic (or toral) part  $\mathcal{C}$  corresponding to the finite cyclic subgroups  $C_n$  and the circle  $T = SO(2)$ .
- The dihedral part  $\mathcal{D}$  corresponding to the finite dihedral subgroups isomorphic to  $D_{2n}$  for some  $n$  and the group  $O(2)$  itself.

The model of the cyclic part is  $\mathcal{A}(SO(2))[W]$  (where the group  $W = O(2)/SO(2)$  acts on the ring  $\mathcal{O}_{\mathcal{F}}$ ). Since all subgroups of  $SO(2)$  are fixed under conjugation by  $W$ ,

$$\mathrm{Spc}(\mathcal{A}(SO(2))[W]) = \mathrm{Spc}(\mathcal{A}(SO(2))) = \{\wp_{C_n} \mid n \geq 1\} \cup \{\wp_{SO(2)}\} =$$


with  $\wp_{C_n} \subset \wp_{SO(2)}$ .

The dihedral part corresponds to  $W$ -equivariant sheaves over the Polish Point  $PP = \{0\} \cup \{1/2n \mid n \geq 1\}$ , where 0 corresponds to the subgroup  $O(2)$  and the point  $1/2n$  corresponds to the conjugacy class of  $D_{2n}$ . The action of  $W$  on the fibre over 0 is trivial. As a topological space, the spectrum of this category is homeomorphic to  $PP$ . Indeed, one sees that whenever a finite complex has  $O(2)$  in its geometric isotropy it has all but finitely many dihedral subgroups.

Altogether

$$\mathrm{Spc}(O(2)\text{-spectra}/\mathbb{Q}) = \{\wp_{C_n} \mid n \geq 1\} \cup \{\wp_{SO(2)}\} \amalg PP$$

9.D. **The rotation group  $SO(3)$ .** The homotopy category of  $SO(3)$ -spectra splits into three parts.

- The cyclic (or toral) part  $\mathcal{C}$  corresponding to the finite cyclic subgroups isomorphic to  $C_n$  for some  $n$  and the maximal tori (each conjugate to  $T = SO(2)$ ).
- The dihedral part  $\mathcal{D}'$  corresponding to the finite dihedral subgroups isomorphic to  $D_{2n}$  for some  $n \geq 2$  and the normalizers of the maximal tori (each conjugate to  $O(2)$ ). The dihedral group of order 2 from  $O(2)$  is conjugate in  $SO(3)$  to the cyclic group of order 2, and so occurs under the first bullet point.
- Three conjugacy classes of exceptional subgroups conjugate to the tetrahedral, octahedral or icosahedral subgroups.

The restriction along  $i : O(2) \longrightarrow SO(3)$  is a map  $\rho : SO(3)\text{-spectra} \longrightarrow O(2)\text{-spectra}$

- $\rho^*$  gives a homeomorphism from the cyclic part of the spectrum of  $SO(3)$ -spectra to the cyclic part of the spectrum of  $O(2)$ -spectra (see Section 10 for more details).

$$\rho^* : \mathrm{Spc}(\mathcal{C}(SO(3))) \xrightarrow{\cong} \mathrm{Spc}(\mathcal{C}(O(2))).$$

- a homeomorphism on the space  $\mathcal{D}'$  of dihedral groups of order  $\geq 4$

$$\rho^* : \mathrm{Spc}(\mathcal{D}(SO(3))) \xrightarrow{\cong} \mathrm{Spc}(\mathcal{D}'(O(2))).$$

- $\rho^*(\wp_{D_2}^{O(2)}) = \wp_{C_2}^{SO(3)}$ .

In other words

$$\mathrm{Spc}(\mathcal{A}(SO(3))) = \mathrm{Spc}(\mathcal{A}(SO(2))) \amalg \mathcal{D}'(O(2)) \amalg \{\wp_{Tet}, \wp_{Oct}, \wp_{Icos}\}.$$

## 10. TORAL $G$ -SPECTRA

Toral  $G$ -spectra are those whose geometric isotropy consists of subgroups of a maximal torus. The point of this restriction is that the analysis can essentially be reduced to the maximal torus. An abelian algebraic model and calculation scheme is given in [14], and in [2] it is shown to give a Quillen equivalence.

We show here that the Balmer spectrum of finite toral  $G$ -spectra is obtained by Going Up and Going Down from that for finite  $T$ -spectra for the maximal torus  $T$ .

**10.A. The landscape.** The category of rational *toral*  $G$ -spectra is a retract of the category of all rational  $G$ -spectra, namely the one obtained from by applying the idempotent  $e_{(T)}$  of the Burnside ring  $A(G) = [S^0, S^0]^G$  corresponding to the subgroups of a maximal torus.

An abelian algebraic model  $\mathcal{A}(G, \text{toral})$  and calculation scheme was given in [14] and it is shown to be Quillen equivalent to rational toral  $G$ -spectra in [2]. Once a monoidal Quillen equivalence has been established in the torus-equivariant case, one expects a monoidal equivalence in the toral case to follow easily. We outline the conclusions here, but readers wanting more details are referred to the source.

The essential ingredients are the maximal torus  $T$  in  $G$ , its normalizer  $N = N_G(T)$  and the action of the Weyl group  $W = N/T$  on  $T$ . One may consider the restriction functors

$$\begin{array}{ccccc} G\text{-spectra} & \xrightarrow{\text{res}_N^G} & N\text{-spectra} & \xrightarrow{\text{res}_T^N} & T\text{-spectra} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}(G, \text{toral}) & \xrightarrow{\theta_*} & \mathcal{A}(N, \text{toral}) & \longrightarrow & \mathcal{A}(T) \end{array}$$

The functor labelled  $\theta_*$  has the character of an extension of scalars functor. The analysis proceeds by first understanding toral  $N$ -spectra, then showing restriction  $\text{res}_N^G$  is faithful.

For toral  $N$ -spectra we proceed as follows.

- A toral  $N$ -spectrum is essentially a  $T$ -spectrum with a homotopical  $W$ -action. The restriction functor  $\text{res}_T^N$  on the toral part just forgets the  $W$ -action. On maps,

$$[X, Y]^N = ([\text{res}_T^N X, \text{res}_T^N Y]^T)^W,$$

(i.e., the  $W$ -invariants of the  $T$ -maps), so the restriction is faithful but not full.

- Correspondingly,  $\mathcal{A}(N, \text{toral}) = \mathcal{A}(T)[W]$  and restriction again just forgets the  $W$ -action. The  $W$ -action on  $\mathcal{A}(T)$  is suitably twisted.

**10.B. Balmer spectra.** We are now ready to describe the Balmer spectra of finite toral  $G$ -spectra.

**Proposition 10.1.** *The restriction functor  $\text{res}_T^N$  from toral  $N$ -spectra to  $T$ -spectra induces a homeomorphism*

$$\text{Spc}((\text{toral} - N\text{-spectra})^c) \cong \text{Spc}(T\text{-spectra}^c)/W \cong \text{Sub}_a(T)/W.$$

*The precisely similar statement holds for algebraic models.*

$$\text{Spc}(D(\mathcal{A}(N, \text{toral}))^c) \cong \text{Spc}(D(\mathcal{A}(T))^c)/W \cong \text{Sub}_a(T)/W.$$



**Remark 10.2.** This is analogous to considering a ring  $R$  with an action of a finite group  $W$  and then considering the ring map  $u : R[W] \longrightarrow R$ . This induces a homeomorphism  $\mathrm{Spc}(R[W]) \cong \mathrm{Spc}(R)/W$ .

We will return to the proof in Subsection 10.C below.

The restriction  $\mathrm{res}_N^G$  from toral  $G$ -spectra to toral  $N$ -spectra turns out to induce a homeomorphism on Balmer spectra of small objects.

**Proposition 10.3.** *The restriction functor  $\mathrm{res}_N^G$  from toral  $G$ -spectra to toral  $N$ -spectra induces a homeomorphism*

$$\mathrm{Spc}((\text{toral} - G\text{-spectra})^c) \cong \mathrm{Spc}((\text{toral} - N\text{-spectra})^c)$$

*The precisely similar statement holds for algebraic models.*

$$\mathrm{Spc}(D(\mathcal{A}(G, \text{toral}))^c) \cong \mathrm{Spc}(D(\mathcal{A}(N, \text{toral}))^c)$$

**Remark 10.4.** This is analogous to considering a ring  $R$  with an action of a finite group  $W$  and then considering the inclusion  $\theta : R^W \longrightarrow R[W]$ . This induces a homeomorphism  $\mathrm{spec}(R^W) = \mathrm{Spc}(R^W) \cong \mathrm{Spc}(R[W])$ .

We will return to the proof in Subsection 10.C below.

We note that  $G$ -conjugacy classes of subgroups of a maximal torus correspond precisely to  $W$ -orbits of subgroups of  $T$ . Thus if we write  $\mathrm{Sub}_a(G, \text{toral})$  for the closed subgroups of a maximal torus of  $G$  we reach the following conclusion.

**Corollary 10.5.** *For a compact Lie group  $G$  with maximal torus  $T$  and  $W = N_G(T)/T$ , we have*

$$\mathrm{Spc}((\text{toral} - G\text{-spectra})^c) = \mathrm{Sub}_a(T)/W = \mathrm{Sub}_a(G, \text{toral})/G$$

$$\mathrm{Spc}(D(\mathcal{A}(G, \text{toral}))^c) = \mathrm{Sub}_a(T)/W = \mathrm{Sub}_a(G, \text{toral})/G. \quad \square$$

**10.C. Toral proofs.** We now prove the results described in Subsection 10.B. One strategy would be to prove this by analogy with a ring  $R$  with  $W$ -action and the sequence of ring maps

$$R^W \longrightarrow R[W] \longrightarrow R$$

corresponding to

$$\text{toral} - G\text{-spectra} \longrightarrow \text{toral} - N\text{-spectra} \longrightarrow T\text{-spectra}.$$

In this view the first map induces a homeomorphism of spectra and the second is the quotient by  $W$ . However we will instead treat the general case of the map

$$\rho : \text{toral} - G\text{-spectra} \longrightarrow T\text{-spectra},$$

and show directly that it is the quotient by  $W$ . The same argument applies in particular when  $G = N$ .

As for tori, we start by working to understand finitely generated thick tensor ideals  $\mathrm{thick}_{\otimes}(X)$ .

**Proposition 10.6.** *If  $X$  is a finite  $G$ -spectrum then  $\text{thick}_\otimes(X)$  only depends on  $\mathcal{I}_g(X)$ . The geometric isotropy  $\mathcal{I}_g(X)$  is a union of conjugacy classes of closed subgroups, it is closed under passage to cotoral subgroups and it has finitely many maximal elements  $(K_1), \dots, (K_r)$ , where we suppose (without loss of generality) that  $K_i \subseteq T$ . In this case*

$$\text{thick}_\otimes(X) = \text{thick}_\otimes(\sigma_{(K_1)}^G, \dots, \sigma_{(K_r)}^G).$$

**Proof:** If  $K \in \mathcal{I}_g(X)$  then from the analysis for tori,

$$\sigma_K = \sigma_K^T \in \text{thick}_\otimes(\text{res}_T^G(X)).$$

This means there is a process for constructing  $\sigma_K$  by use of cofibre sequences, retracts and tensoring with finite  $T$ -spectra. We will apply coinduction to this process. Some care is necessary for smashing with a spectrum.

**Lemma 10.7.** (a) *If  $X$  is a  $G$ -spectrum then*

$$F_T(G_+, \text{res}_T^G(X) \wedge Y) \simeq X \wedge F_T(G_+, Y)$$

(b) *Coinduction is monoidal on toral spectra: the unit map  $S^0 \longrightarrow F_N(G_+, S^0)$  is an equivalence on toral parts, and for toral  $N$ -spectra  $A$  and  $B$  the natural map*

$$F_N(G_+, A) \wedge F_N(G_+, B) \longrightarrow F_N(G_+, A \wedge B)$$

*of toral  $G$ -spectra is an equivalence.*

**Proof:** Part (a) is the well known Frobenius reciprocity property.

Part (b) follows from the statement that the unit

$$X \longrightarrow F_N(G_+, \text{res}_N^G X)$$

is an equivalence of toral spectra [14, Subsection 10.A, or Corollary 7.11 for the algebraic model]. At the very core, it is the statement that for any compact Lie group  $G/N$  has trivial rational homology.  $\square$

**Corollary 10.8.** *If  $Z \in \text{thick}_\otimes(A)$  then  $F_T(G_+, Z) \in \text{thick}_\otimes(F_T(G_+, A))$ .*

**Proof:** Coinduction preserves triangles and retracts, so the only point is to deal with smash products.

We proceed in two steps. From  $T$  to  $N$  we use the fact that every  $T$ -spectrum  $A$  is the restriction of an  $N$ -spectrum  $\tilde{A}$  (for example we may take  $\tilde{A}$  to have ‘trivial  $W$ -action’ obtained by inducing up  $A$  and then taking the idempotent summand corresponding to the trivial representation). Hence we may apply Lemma 10.7 (a) to see

$$F_T(N_+, A \wedge B) \simeq \tilde{A} \wedge F_T(N_+, B).$$

Going from  $N$  to  $G$  we use the monoidal property of Lemma 10.7 (b).  $\square$

Now, since  $\sigma_K \in \text{thick}_\otimes(\text{res}_T^G X)$  we conclude from Corollary 10.8 that

$$F_T(G, \sigma_K) \in \text{thick}_\otimes(F_T(G_+, \text{res}_T^G(X))) = \text{thick}_\otimes(F_T(G_+, S^0) \wedge X) \subseteq \text{thick}_\otimes(X).$$

Now  $\sigma_{(K)}^N$  is a retract of  $F_T(N_+, \sigma_K)$ , and hence  $\sigma_{(K)}^G$  is a retract of  $F_T(G_+, \sigma_K)$  and hence in  $\text{thick}_\otimes(X)$ .

By considering  $X$  as a  $T$ -spectrum we see,  $\mathcal{I}_g(X)$  has only finitely many maximal conjugacy classes  $(K_1), \dots, (K_r)$ . Thus

$$\text{thick}_{\otimes}(\sigma_{(K_1)}, \dots, \sigma_{(K_r)}) \subseteq \text{thick}_{\otimes}(X).$$

From the case of the torus we see

$$\text{res}_T^G(X) \in \text{thick}_{\otimes}(\text{res}_T^G \sigma_{(K_1)}, \dots, \text{res}_T^G \sigma_{(K_r)}),$$

and applying coinduction

$$X \in \text{thick}_{\otimes}(\sigma_{(K_1)}, \dots, \sigma_{(K_r)}).$$

□

Finally, we turn to the statement about primes, continuing to write

$$\rho : G\text{-spectra} \longrightarrow T\text{-spectra}$$

for restriction.

**Proof:** It is clear that the pullbacks of  $\wp_K$  and  $\wp_{K'}$  are the same if  $K$  is  $G$ -conjugate to  $K'$ . Finally for injectivity, it is clear that if  $K$  is not conjugate to  $K'$  then if  $\dim(K) \geq \dim(K')$  then

$$\sigma_{(K)} \in \rho^*(\wp_{K'}) \setminus \rho^*(\wp_K).$$

Hence  $\rho$  induces an injective map

$$\bar{\rho} : \text{Sub}_a(T)/W \longrightarrow \text{Spc}((\text{toral} - G\text{-spectra})^c).$$

To see it is surjective, we repeat the proof of Theorem 7.2, but with subgroups replaced by conjugacy classes. □

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